Convergence of the Euler Scheme for a Class of Stochastic Differential Equation

Glenn MARION†, Xuerong MAO, Eric RENSHAW

Department of Statistics and Modelling Science, Livingstone Tower, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK.
† contact email address: glenn@stams.strath.ac.uk

Abstract: Stochastic differential equations provide a useful means of introducing stochasticity into models across a broad range of systems from chemistry to population biology. However, in many applications the resulting equations have so far proved intractable to direct analytical solution. Numerical approximations, such as the Euler scheme, are therefore a vital tool in exploring model behaviour. Unfortunately, current results concerning the convergence of such schemes impose conditions on the drift and diffusion coefficients of the stochastic differential equation, namely the linear growth and global Lipschitz conditions, which are often not met by systems of interest. In this paper we relax these conditions and prove that numerical solutions based on the Euler scheme will converge to the true solution of a broad class of stochastic differential equations. The results are illustrated by application to a stochastic Lotka-Volterra model and a model of chemical auto-catalysis, neither of which satisfy either the linear growth nor the global Lipschitz conditions.

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1 Introduction

Consider stochastic differential equations (S.D.E.s) of the form

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{1.1}$$

where

$$x = (x_1, ..., x_n)^T \ , \ f(x) = (f_1(x), ..., f_n(x))^T \ , \ g(x) = (g_{ij}(x))_{n\times m}$$

and $B(t)$ is an m-dimensional Brownian motion defined on a given complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). The solution to (1.1), $x(t)$, is a member of the open set $G \subseteq R^n$ for $t \in [0, T]$ and initial value $x_0 \in G$. That is, G is an invariant set of equation (1.1). We note that G is often the positive cone $R^n_+ \equiv \{x \in R^n : x > 0\}$ or the whole Euclidean space R^n. The solution $x(t)$ is to be interpreted as the stochastic integral

$$x(t) = x_0 + \int_0^t f(x(s))ds + \int_0^t g(x(s))dB(s), \tag{1.2}$$

in the Itô sense. Equation (1.1) represents one method in which stochasticity may be introduced into the deterministic model

$$\dot{x}(t) = f(x(t)).$$

In general, system (1.1) is analytically intractable (see for example Mao, 1997), and therefore numerical approximation schemes such as the Euler (or Euler-Maruyama) approximation are invaluable tools for exploring its properties. Most existing proofs of the convergence of the such numerical schemes (Kloeden and Platen, 1992; Mao, 1997; Skokorod 1965) rely on the following conditions:
(Linear growth) there exists a positive constant L such that

$$\mid f(x) \mid^2 \vee \mid g(x) \mid^2 \leq L(1+ \mid x \mid^2); \tag{1.3}$$

and, *(Global Lipschitz)* there exists a positive constant $\bar{L}$ such that

$$\mid f(x) - f(y) \mid^2 \vee \mid g(x) - g(y) \mid^2 \leq \bar{L} \mid x - y \mid^2 \tag{1.4}$$

For all $x \in R^n$.

Unfortunately these conditions are often not met by systems of interest. For example, consider the stochastic Lotka-Volterra model

$$dx(t) = \text{diag}(x_1, ..., x_n(t)) \left[A(x(t) - \bar{x})dt + \sigma(x(t) - \bar{x})dB(t)\right]; \tag{1.5}$$

where

$$x(t) = (x_1, ..., x_n(t))^T \ , \ \bar{x} = (\bar{x}_1, ..., \bar{x}_n)^T \ , \ A = (a_{ij})_{n\times n} \ , \ \sigma = (\sigma_{ij})_{n\times n} \ ,$$

and $B(t)$ is now a scalar Brownian motion. This is of the form (1.1) with $f(x) = \mathrm{diag}(x_1, ..., x_n)A(x - \bar{x})$ and $g(x) = \mathrm{diag}(x_1, ..., x_n)\sigma(x - \bar{x})$. It is straightforward to see that neither the linear growth condition nor the global Lipschitz condition will be satisfied by this system. The properties of the solution $x(t) = x(t; x_0)$ to equation (1.5) are discussed in detail by Mao, Marion and Renshaw (2000). Here we discuss numerical approximations to a class of S.D.E.s including this system, but note that our approach developed from initial studies of the numerical solutions of (1.5). In particular, we show that under certain conditions, weaker than (1.3) and (1.4), the Euler scheme applied to system (1.1), converges to the analytic solution $x(t)$, and in doing so bound the order of this approximation. We note that others have also weakened these conditions. Yamada (1978) relaxed the global Lipschitz condition, whilst Kaneko and Nakao (1988) have shown that the Euler approximation converges, in the strong sense, to the solution of the stochastic differential equation whenever path-wise uniqueness of the solution holds. However, both results require the linear growth condition whilst the latter provides no information on the order of approximation.

In Section 2 the Euler approximation is introduced and we state Theorem 1, namely that, under certain conditions, the Euler approximate solution converges to the true solution for system (1.1). This result is proved in Sections 3, 4 and 5. Section 3 demonstrates that the Euler approximate solution will converge to the true solution $x(t)$ if both remain within a compact set. A compact set appropriate to our needs is introduced in Section 4; the escape times from this set, both of $x(t)$ and the approximate solution, are bounded in probability. These results are brought together in Section 5 to prove our convergence result of Theorem 1. Finally, Section 6 applies this result to two interesting models, namely the Lotka-Volterra system (1.5) and a model describing an auto-catalytic reaction system.

2 The Euler Approximation

For system (1.1) the discrete time Euler approximation on $t \in \{0, \Delta t, ..., N\Delta t = T\}$ is given by the iterative scheme

$$x_{\Delta t}(t + \Delta t) = x_{\Delta t}(t) + f(x_{\Delta t}(t))\Delta t + g(x_{\Delta t}(t))\Delta B(t) \tag{2.1}$$

with initial value $x_{\Delta t}(0) = x_0$. Here the time increment is Δt, and the $\Delta B(t) = B(t + \Delta t) - B(t)$ represent N independent draws from an m-dimensional Normal distribution whose individual components have mean zero and variance

Δt. We will prove the following useful convergence result.

Theorem 1 *Let G be an open subset of R^n, and denote the unique solution of (1.1) for $t \in [0,T]$ given $x_0 \in G$ by $x(t) \in G$. Define $x_{\Delta t}(t)$ as the Euler approximation (2.1) and let $\mathcal{D} \subseteq G$ be any compact set. Suppose the following conditions are satisfied:*

(i) (local Lipschitz condition) there exists a positive constant $K_1(\mathcal{D})$ such that $x, y \in \mathcal{D}$

$$| f(x) - f(y) |^2 \vee | g(x) - g(y) |^2 \leq K_1(\mathcal{D}) \mid x - y \mid^2;$$

(ii) there exists a C^2-function $V : G \to R_+$ such that $\{x \in G : V(x) \leq r\}$ is compact for any $r > 0$;

(iii) $LV(x) \leq K(1 + V(x))$ where

$$LV(x) \equiv V_x(x) f(x) + \frac{1}{2} trace \left[g^T(x) V_{xx}(x) g(x) \right]$$

is the diffusion operator associated with (1.1);

(iv) there exists a positive constant $K_3(\mathcal{D})$ such that for all $x, y \in \mathcal{D}$

$$| V(x) - V(y) | \vee | V_x(x) - V_x(y) | \vee | V_{xx}(x) - V_{xx}(y) | \leq K_3(\mathcal{D}) \mid x - y \mid .$$

Then for any $\epsilon, \delta > 0$ there exists $\Delta t^* > 0$ such that

$$\Pr \left(\sup_{0 \leq t \leq T} | x_{\Delta t}(t) - x(t) |^2 \geq \delta \right) \leq \epsilon,$$

provided $\Delta t \leq \Delta t^*$ and the initial value $x_0 \in G$.

In order to prove this result it is useful to note that if condition (i) holds then there exists a positive constant $K_2(\mathcal{D})$ such that for $x, y \in \mathcal{D}$

$$| f(x) |^2 \vee | g(x) |^2 \leq K_2(\mathcal{D}). \tag{2.2}$$

Furthermore, we shall rewrite $x_{\Delta t}(t)$ as the integral

$$x_{\Delta t}(t) = x_0 + \int_0^t f(\hat{x}_{\Delta t}(s)) ds + \int_0^t g(\hat{x}_{\Delta t}(s)) dB(s), \tag{2.3}$$

where we have introduced the piecewise constant process

$$\hat{x}_{\Delta t}(t) = \Sigma_{k=1}^N x_{\Delta t}((k-1)\Delta t) I_{[(k-1)\Delta t, k\Delta t]}(t), \tag{2.4}$$

and I_A is the indicator function for set A. Expression (2.3) extends the definition of the Euler scheme to all $t \in [0, T]$, and may also be expressed in the stochastic differential form

$$dx_{\Delta t}(t) = f(\hat{x}_{\Delta t}(t))dt + g(\hat{x}_{\Delta t}(t))dB(t), \tag{2.5}$$

with initial condition $x_{\Delta t}(0) = x_0 \in G$.

3 Convergence of the Euler scheme

To proceed we consider only trajectories $x(t)$ and $x_{\Delta t}(t)$ which remain within a bounded region $\mathcal{D}$. To achieve this, introduce the stopping time $\tau = \rho \wedge \theta$ where

$$\rho = \inf\{t \geq 0 : x_{\Delta t}(t) \notin \mathcal{D}\} \qquad \text{and} \qquad \theta = \inf\{t \geq 0 : x(t) \notin \mathcal{D}\}$$

are the first times that $x_{\Delta t}(t)$ and $x(t)$, respectively, leave $\mathcal{D}$. We will define $\mathcal{D}$ more precisely later.

Let $T_1 \in [0, T]$ be an arbitrary time. Then subtracting (1.2) from (2.3), applying the inequality $|\, a + b\,|^2 \leq 2\,|\, a\,|^2 + 2\,|\, b\,|^2$, taking the supremum over $t \in [0, \tau \wedge T_1]$, and then finally the expectation, leads to

$$\begin{aligned} \mathrm{E}\left[\sup_{0\leq t\leq \tau\wedge T_1} |\, x_{\Delta t}(t) - x(t)\,|^2\right] \leq\ & 2\mathrm{E} \sup_{0\leq t\leq \tau\wedge T_1} |\int_0^t [f(\hat{x}_{\Delta t}(s) - f(x(s))]\, ds\,|^2 \\ & + 2\mathrm{E} \sup_{0\leq t\leq \tau\wedge T} |\int_0^t [g(\hat{x}_{\Delta t}(s)) - g(x(s))]\, dB(s)\,|^2 . \end{aligned} \tag{3.1}$$

The Hölder inequality shows that

$$2\mathrm{E} \sup_{0\leq t\leq \tau\wedge T_1} |\int_0^t [f(\hat{x}_{\Delta t}(s)) - f(x(s))]ds\,|^2 \leq 2T\ \mathrm{E}\int_0^{\tau\wedge T_1} |\, f(\hat{x}_{\Delta t}(s) - f(x(s))\,|^2\, ds, \tag{3.2}$$

whence applying the well-known Doob inequality (c.f. Theorem 1.7.2, Mao, 1997) to the second term of (3.1) leads to

$$2\mathrm{E} \sup_{0\leq t\leq \tau\wedge T_1} |\int_0^t [g(\hat{x}_{\Delta t}(s)) - g(x(s))]dB(s)\,|^2 \leq 8\,\mathrm{E}\int_0^{\tau\wedge T_1} |\, g(\hat{x}_{\Delta t}(s)) - g(x(s))\,|^2\, ds. \tag{3.3}$$

If the coefficients of (1.1) are locally Lipschitz continuous (i.e. satisfy condition (i) of Theorem 1), then since both $x(s)$ and $x_{\Delta t}(s)$ are bounded we may write

$$|\, f(\hat{x}_{\Delta t}(s) - f(x(s))\,|^2 \vee |\, g(\hat{x}_{\Delta t}(s) - g(x(s))\,|^2 \leq K_1(\mathcal{D})\,|\,\hat{x}_{\Delta t}(s) - x(s)\,|^2 \tag{3.4}$$

for $s \in [0, \tau \wedge T_1]$. Substituting (3.2), (3.3) and (3.4) into (3.1) then reveals that

$$
\begin{aligned}
&\mathrm{E}\left[\sup_{0\le t\le \tau\wedge T_1} \mid x_{\Delta t}(t) - x(t) \mid^2\right] \\
&\le\ 2K_1(\mathcal{D})(T+4)\ \mathrm{E}\int_0^{\tau\wedge T_1} \mid \hat{x}_{\Delta t}(s) - x(s) \mid^2 ds \\
&=\ 2K_1(\mathcal{D})(T+4)\ \mathrm{E}\int_0^{\tau\wedge T_1} \mid \hat{x}_{\Delta t}(s) - x_{\Delta t}(s) + x_{\Delta t}(s) - x(s) \mid^2 ds \\
&\le\ 4K_1(\mathcal{D})(T+4)\ \mathrm{E}\int_0^{\tau\wedge T_1} \Big(\mid \hat{x}_{\Delta t}(s) - x_{\Delta t}(s) \mid^2 + \mid x_{\Delta t}(s) - x(s) \mid^2\Big)\, ds \\
&\le\ 4K_1(\mathcal{D})(T+4)\ \mathrm{E}\int_0^{\tau\wedge T_1} \mid \hat{x}_{\Delta t}(s) - x_{\Delta t}(s) \mid^2 ds \\
&\qquad +4K_1(\mathcal{D})(T+4)\int_0^{T_1}\ \mathrm{E}\left[\sup_{0\le s'\le \tau\wedge s} \mid x_{\Delta t}(s') - x(s') \mid^2\right] ds. \qquad (3.5)
\end{aligned}
$$

Bounding the first term on the right-hand side of (3.5) and then applying the Gronwall inequality leads to a bound on $\mathrm{E}\left[\sup_{0\le t\le \tau\wedge T} \mid x_{\Delta t}(t) - x(t) \mid^2\right]$. Inspection of (2.4) reveals that $\hat{x}_{\Delta t}(s) = x_{\Delta t}([s/\Delta t]\Delta t)$ where $[s/\Delta t]$ is the integer part of $s/\Delta t$. We can now use (2.3) to show that

$$
\begin{aligned}
&\mid \hat{x}_{\Delta t}(s) - x_{\Delta t}(s) \mid^2 = \mid x_{\Delta t}([s/\Delta t]\Delta t) - x_{\Delta t}(s) \mid^2 \\
&= \mid \int_{[s/\Delta t]\Delta t}^{s} f(x_{\Delta t}([s/\Delta t]\Delta t))du + \int_{[s/\Delta t]\Delta t}^{s} g(x_{\Delta t}([s/\Delta t]\Delta t))dB(u) \mid^2 \\
&\le 2 \mid f(x_{\Delta t}([s/\Delta t]\Delta t)) \mid^2 \Delta t^2 + 2 \mid g(x_{\Delta t}([s/\Delta t]\Delta t)) \mid^2 \mid B(s) - B([s/\Delta t]\Delta t) \mid^2 \\
&\le 2K_2(\mathcal{D})\Delta t^2 + 2K_2(\mathcal{D}) \mid B(s) - B([s/\Delta t]\Delta t) \mid^2 .
\end{aligned}
$$

Note that the last line follows provided $s \in [0, \tau\wedge T_1]$ and condition (2.2) holds. If $T\Delta t < 1$ this inequality leads to

$$
\begin{aligned}
\mathrm{E}\int_0^{\tau\wedge T_1} \mid \hat{x}_{\Delta t}(s) - x_{\Delta t}(s) \mid^2 ds \ &\le\ 2K_2(\mathcal{D})T_1\Delta t^2 \\
&\qquad +2K_2(\mathcal{D})\int_0^{T_1} \mathrm{E} \mid B(s) - B([s/\Delta t]\Delta t) \mid^2 ds \\
&\le\ 2K_2(\mathcal{D})T\Delta t^2 + 2K_2(\mathcal{D})T_1 m\Delta t \\
&\le\ 2K_2(\mathcal{D})(mT+1)\Delta t \qquad (3.6)
\end{aligned}
$$

(recall that our Brownian motion has dimension m). Using this result in (3.5) shows that

$$
\begin{aligned}
\mathrm{E}\left[\sup_{0\le t\le \tau\wedge T_1} \mid x_{\Delta t}(t) - x(t) \mid^2\right] &\le C_1(\mathcal{D})\Delta t \\
&\qquad + C_2(\mathcal{D})\int_0^{T_1} \mathrm{E}\left[\sup_{0\le r\le \tau\wedge s} \mid x_{\Delta t}(r) - x(r) \mid^2\right] ds
\end{aligned}
$$

where $C_1(\mathcal{D}) = 8K_1(\mathcal{D})K_2(\mathcal{D})(T+4)(mT+1)$ and $C_2(\mathcal{D}) = 4K_1(\mathcal{D})(T+4)$. On applying the Gronwall inequality we then have the following theorem.

Theorem 2 *If τ is the first exit time of either the solution $x(t)$ or the Euler approximate solution $x_{\Delta t}(t)$ from a bounded region $\mathcal{D}$, and $f(x)$ and $g(x)$ satisfy condition (i) of Theorem 1, then for $\Delta t T < 1$*

$$\mathrm{E}\left[\sup_{0\le t\le \tau\wedge T} \mid x_{\Delta t}(t) - x(t) \mid^2\right] \le C_1(\mathcal{D})e^{C_2(\mathcal{D})T}\Delta t = C(\mathcal{D})\Delta t.$$

Thus, as long as $x_{\Delta t}(t)$ and $x(t)$ remain in $\mathcal{D}$ the Euler scheme $x_{\Delta t}(t)$ converges to the solution $x(t)$ of equation (1.1) as $\Delta t \to 0$.

4 Characterising stopping times

To proceed further we define the bounded domain

$$\mathcal{D} = \mathcal{D}(r) \equiv \{x \in G \text{ such that } V(x) \le r\}.$$

In order to prove Theorem 1 we will determine the probability that both $x_{\Delta t}(t)$ and $x(t)$ remain in $\mathcal{D}(r)$. To do so we assume the existence of the non-negative function $V(x)$ satisfying condition (ii) of Theorem 1. Since $x(t)$ is governed by equation (1.1), applying Itô's formula to $V(x(t))$ yields

$$dV(x(t)) = LV(x(t)) + V_x(x(t))g(x(t))dB(t).$$

Integrating from 0 to $t\wedge\theta$ and taking expectations gives

$$\mathrm{E}[V(x(t\wedge\theta)] = V(x_0) + \mathrm{E}\int_0^{t\wedge\theta} LV(x(s))ds.$$

Whence applying condition (iii) of Theorem 1 leads to

$$\begin{aligned} \mathrm{E}[V(x(t\wedge\theta)] &\le V(x_0) + K\mathrm{E}\int_0^{t\wedge\theta}(1+V(x(s)))ds. \\ &\le (V(x_0)+KT) + \int_0^t \mathrm{E}[V(x(s\wedge\theta))]ds \\ &\le (V(x_0)+KT)e^{KT}. \end{aligned}$$

The last line follows on application of the Gronwall inequality. On noting that $V(x(\theta)) = r$, since $x(\theta)$ is on the boundary of $\mathcal{D}(r)$, the probability $p(\theta < T)$ can now be bounded as follows.

$$\begin{aligned} (V(x_0)+KT)e^{KT} &\ge \mathrm{E}[V(x(t\wedge\theta)] \\ &\ge \mathrm{E}[V(x(\theta))I_{\{\theta<T\}}(\omega)] \\ &\ge r\mathrm{E}[I_{\{\theta<T\}}(\omega)] \\ &\ge rP(\theta<T), \end{aligned} \tag{4.1}$$

whence rearranging (4.1) leads to

$$P(\theta < T) \leq (V(x_0) + KT)e^{KT}/r = \tilde{\epsilon}. \tag{4.2}$$

Here r can be made as large as required, for a given T and x_0, to accommodate any $\tilde{\epsilon} \in (0, 1)$. Theorem 3 now follows immediately.

Theorem 3 *If θ is the first exit time of the solution $x(t)$ to equation (1.1) from the domain $\mathcal{D}(r)$, and a function $V(x)$ exists which satisfies conditions (ii) and (iii) of Theorem 1, then*

$$\mathrm{P}(\theta \geq T) \geq 1 - \tilde{\epsilon}.$$

We note that the following useful result follows directly from Theorem 3.

Lemma 4 *Let θ be the first exit time of the solution $x(t)$ to equation (1.1) from the domain $\mathcal{D}(r)$, and let the coefficients of (1.1) satisfy condition (i) of Theorem 1. If a function $V(x)$ exists which satisfies conditions (ii) and (iii) of Theorem 1, then the limit of $\lim_{r\to\infty} D(r) \equiv G$ and, for $t \in [0, T]$ and $x_0 \in G$, $x(t)$ remains in G. Furthermore, $x(t)$ is the unique solution of equation (1.1) on $t \in [0, T]$ for all finite T.*

In order to prove this, first note that if the coefficients of (1.1) are locally Lipschitz continuous then there exists a unique local solution $x(t)$ on $t \in [0, \tau_e]$ where τ_e is some random explosion time (cf. Arnold, 1972 or Mao, 1997). Since $D(r)$ is an increasing set (with r) $\lim_{r\to\infty} D(r) = \bigcup_{r=1}^{\infty} D(r) \subseteq G$. Suppose that $\lim_{r\to\infty} D(r) \not\equiv G$, then there exists some $x \in G$ such that $x \notin \bigcup_{r=1}^{\infty} D(r)$. However, if $x \in G$ then $V(x) < \infty$ and there exists an $r > 0$ such that $V(x) < r$, which implies that $x \in \bigcup_{r=1}^{\infty} D(r)$ and therefore that $\lim_{r\to\infty} D(r) \equiv G$. Now by Theorem 3 the probability of escape from the set $\lim_{r\to\infty} D(r)$ in any finite T is zero. Thus $x(t)$ must remain in G for any finite time, and $\lim_{r\to\infty} D(r)$ identifies the invariant set $G \subseteq R^n$ of Theorem 1. Since $G \subseteq R^n$, this implies there will be no explosion in any finite time, and so $x(t)$ is unique on any finite interval $t \in [0, T]$.

We require a similar result to Theorem 3 for the Euler approximate solutions $x_{\Delta t}(t)$. Noting that $x_{\Delta t}(t)$ is the solution to (2.5), and applying the Itô formula to $V(x_{\Delta t}(t))$ yields,

$$\begin{aligned} dV(x_{\Delta t}(t)) &= \left[V_x(x_{\Delta t}(t))f(\hat{x}_{\Delta t}(t)) + \frac{1}{2}g^T(\hat{x}_{\Delta t}(t))V_{xx}(x_{\Delta t}(t))g(\hat{x}_{\Delta t}(t))\right]dt \\ &\quad + V_x(x_{\Delta t}(t))g(\hat{x}_{\Delta t}(t))dB(t) \\ &= LV(\hat{x}_{\Delta t}(t)) + \left[V_x(x_{\Delta t}(t)) - V_x(\hat{x}_{\Delta t}(t))\right]f(\hat{x}_{\Delta t}(t))dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}g^T(\hat{x}_{\Delta t}(t))\left[V_{xx}(x_{\Delta t}(t))-V_{xx}(\hat{x}_{\Delta t}(t))\right]g(\hat{x}_{\Delta t}(t))dt \\
&+V_x(x_{\Delta t}(t))g(\hat{x}_{\Delta t}(t))dB(t).
\end{aligned}
$$

Whence on applying condition (iii) of Theorem 1 we obtain

$$
\begin{aligned}
dV(x_{\Delta t}(t)) \quad \le \quad & K\left(1+V(\hat{x}_{\Delta t}(t))\right)+\left[V_x(x_{\Delta t}(t))-V_x(\hat{x}_{\Delta t}(t))\right]f(\hat{x}_{\Delta t}(t))dt \\
&+\frac{1}{2}g^T(\hat{x}_{\Delta t}(t))\left[V_{xx}(x_{\Delta t}(t))-V_{xx}(\hat{x}_{\Delta t}(t))\right]g(\hat{x}_{\Delta t}(t))dt \\
&+V_x(x_{\Delta t}(t))g(\hat{x}_{\Delta t}(t))dB(t) \\
\le \quad & K\left(1+V(x_{\Delta t}(t))\right)dt+K\left[V(\hat{x}_{\Delta t}(t))-V(x_{\Delta t}(t))\right]dt \\
&+\left[V_x(x_{\Delta t}(t))-V_x(\hat{x}_{\Delta t}(t))\right]f(\hat{x}_{\Delta t}(t))dt \\
&+\frac{1}{2}g^T(\hat{x}_{\Delta t}(t))\left[V_{xx}(x_{\Delta t}(t))-V_{xx}(\hat{x}_{\Delta t}(t))\right]g(\hat{x}_{\Delta t}(t))dt \\
&+V_x(x_{\Delta t}(t))g(\hat{x}_{\Delta t}(t))dB(t).
\end{aligned}
$$

Integrating from 0 to $\rho\wedge t$ and taking expectations gives

$$
\begin{aligned}
\mathrm{E}[V(x_{\Delta t}(\rho\wedge t))] \le\; & V(x_0)+K\mathrm{E}\int_0^{\rho\wedge t}\left(1+V(x_{\Delta t}(s))\right)ds \\
&+K\mathrm{E}\int_0^{\rho\wedge t}\left[V(\hat{x}_{\Delta t}(s))-V(x_{\Delta t}(s))\right]ds \\
&+\mathrm{E}\int_0^{\rho\wedge t}\left[V_x(x_{\Delta t}(s))-V_x(\hat{x}_{\Delta t}(s))\right]f(\hat{x}_{\Delta t}(s))ds \\
&+\frac{1}{2}\mathrm{E}\int_0^{\rho\wedge t}g^T(\hat{x}_{\Delta t}(s))\left[V_{xx}(x_{\Delta t}(s))-V_{xx}(\hat{x}_{\Delta t}(s))\right]g(\hat{x}_{\Delta t}(s))ds \\
\le\; & V(x_0)+KT+K\int_0^t \mathrm{E}[V(x_{\Delta t}(s\wedge\rho))]ds \\
&+K\mathrm{E}\int_0^{\rho\wedge t}\mid V(\hat{x}_{\Delta t}(s))-V(x_{\Delta t}(s))\mid ds \\
&+\mathrm{E}\int_0^{\rho\wedge t}\mid V_x(x_{\Delta t}(s))-V_x(\hat{x}_{\Delta t}(s))\mid\mid f(\hat{x}_{\Delta t}(s))\mid ds \\
&+\frac{1}{2}\mathrm{E}\int_0^{\rho\wedge t}\mid g(\hat{x}_{\Delta t}(s))\mid^2\mid V_{xx}(x_{\Delta t}(s))-V_{xx}(\hat{x}_{\Delta t}(s))\mid ds,
\end{aligned}
$$

and invoking (2.2) and condition (iv) of Theorem 1 leads to

$$
\begin{aligned}
\mathrm{E}[V(x_{\Delta t}(\rho\wedge t))] \quad \le \quad & V(x_0)+KT+\Big(K+K_2^{1/2}(\mathcal{D})+K_2(\mathcal{D})/2\Big) \\
&\times K_3(\mathcal{D})\int_0^{\rho\wedge t}\mathrm{E}\mid\hat{x}_{\Delta t}(s)-x_{\Delta t}(s)\mid ds \\
&+K\int_0^t \mathrm{E}[V(x_{\Delta t}(\rho\wedge s))]ds.
\end{aligned}
$$

The bound

$$
\int_0^{\rho\wedge t}\mathrm{E}\mid\hat{x}_{\Delta t}(s)-x_{\Delta t}(s)\mid ds \le \left(2K_2(\mathcal{D})T(mT+1)\right)^{1/2}\Delta t^{1/2},
$$

follows from Hölders inequality and equation (3.6) for $t \in [0,T]$ and $\Delta t T < 1$. Thus

$$\begin{aligned}
\mathrm{E}[V(x_{\Delta t}(\rho \wedge t))] &\leq V(x_0) + KT + \left(K + K_2^{1/2}(\mathcal{D}) + K_2(\mathcal{D})/2\right) \\
&\quad \times K_3(\mathcal{D}) \left(2K_2(\mathcal{D})T(mT+1)\right)^{1/2} \Delta t^{1/2} \\
&\quad + K \int_0^t \mathrm{E}[V(x_{\Delta t}(s \wedge \rho))] ds.
\end{aligned}$$

Whence, on applying the Gronwall inequality,

$$\mathrm{E}[V(x_{\Delta t}(\rho \wedge T))] \leq \left(V(x_0) + KT\right) e^{KT} + H(\mathcal{D}) \Delta t^{1/2},$$

where $H(\mathcal{D}) = e^{KT}(K + K_2^{1/2}(\mathcal{D}) + K_2(\mathcal{D})/2) K_3(\mathcal{D}) \left(2K_2(\mathcal{D})T(mT+1)\right)^{1/2}$.

An argument analogous to that used to prove Theorem 3 can now be used to bound $P(\rho < T)$. Since $x_{\Delta t}(\rho)$ is on the boundary of $\mathcal{D}(r)$ then $V(x_{\Delta t}(\rho)) = r$ which leads to

$$\begin{aligned}
\left(V(x_0) + KT\right) e^{KT} + H(\mathcal{D}) \Delta t^{1/2} &\geq \mathrm{E}[V(x(\rho \wedge T))] \\
&\geq \mathrm{E}[V(x(\rho)) I_{\{\rho < T\}}(\omega)] \\
&\geq r \mathrm{E}[I_{\{\rho < T\}}(\omega)] \\
&\geq r P(\rho < T).
\end{aligned}$$

Rearranging this inequality and defining $\bar{H}(\mathcal{D}) = H(\mathcal{D}) e^{-KT} / (V(x_0) + KT)$ reveals that

$$P(\rho < T) \leq \tilde{\epsilon} \left(1 + \bar{H}(\mathcal{D}) \Delta t^{1/2}\right),$$

where $\tilde{\epsilon}$ is defined in equation (4.2). This proves the following theorem.

Theorem 5 *Let ρ be the first exit time of the Euler approximate solution (2.3) from the domain $\mathcal{D}(r)$. Then if $f(x)$ and $g(x)$ satisfy condition (i) of Theorem 1 and there exists a function $V(x)$ which satisfies conditions (ii)-(iv) of Theorem 1 then (for sufficiently small Δt)*

$$\mathrm{P}(\rho \geq T) \geq 1 - \tilde{\epsilon} \left(1 + \bar{H}(\mathcal{D}) \Delta t^{1/2}\right).$$

The significance of Theorems 3 and 5 is that both $x(t)$ and $x_{\Delta t}(t)$ remain within the domain $\mathcal{D}(r)$, and therefore by Theorem 2 the Euler scheme will converge to the solution $x(t)$, with probability

$$\mathrm{P}(\tau < T) \leq \mathrm{P}(\rho < T) + \mathrm{P}(s < T) \leq \tilde{\epsilon} \left(2 + \bar{H}(\mathcal{D}) \Delta t^{1/2}\right). \tag{4.3}$$

To prove Theorem 1 we will pursue this argument more rigorously.

5 Convergence in probability

Introducing the event sub-space

$$\bar{\Omega} = \{\omega : \sup_{0 \le t \le T} \mid x_{\Delta t}(t) - x(t) \mid^2 \ge \delta\},$$

and using Theorem 2, we find that

$$\begin{aligned}
C(\mathcal{D})\Delta t &\ge \mathrm{E}\left[\sup_{0 \le t \le \tau \wedge T} \mid x_{\Delta t}(t) - x(t) \mid^2\right] \\
&\ge \mathrm{E}\left[I_{\{\tau \ge T\}}(\omega) \sup_{0 \le t \le T} \mid x_{\Delta t}(t) - x(t) \mid^2\right] \\
&\ge \mathrm{E}\left[I_{\{\tau \ge T\}}(\omega) I_{\{\bar{\Omega}\}}(\omega) \sup_{0 \le t \le T} \mid x_{\Delta t}(t) - x(t) \mid^2\right] \\
&\ge \delta \ \mathrm{E}\left[I_{\{\tau \ge T\}}(\omega) I_{\{\bar{\Omega}\}}(\omega)\right] \\
&= \delta \ \mathrm{P}\left((\tau \ge T) \cap \bar{\Omega}\right) \\
&\ge \delta \left[\ \mathrm{P}\left(\bar{\Omega}\right) - \mathrm{P}\left(\tau < T\right)\right].
\end{aligned}$$

Whence on using (4.3) we conclude that

$$\mathrm{P}\left(\bar{\Omega}\right) = \mathrm{P}\left(\sup_{0 \le t \le T} \mid x_{\Delta t}(t) - x(t) \mid^2 \ge \delta\right) \ \le \ 2\tilde{\epsilon} + \tilde{\epsilon}\bar{H}(\mathcal{D})\Delta t^{1/2} + \frac{C(\mathcal{D})}{\delta}\Delta t$$

which, for appropriate choice of Δt, proves Theorem 1. It is clear that to increase the precision δ then Δt should be reduced. However, it is interesting to note that to leading order the bound on $\mathrm{P}\left(\bar{\Omega}\right)$ is independent of δ. Furthermore, to achieve a given precision with increased probability, Δt should also be reduced. This is because the domain $\mathcal{D}(r)$ grows as $\tilde{\epsilon}$ is reduced (recall that $r \propto 1/\tilde{\epsilon}$), and in general $\bar{H}(\mathcal{D})$ and $C(\mathcal{D})$ will increase with the size of the domain.

6 Two particular models

6.1 Lotka-Volterra model

For the Lotka-Volterra system (1.5) the functions $f(x)$ and $g(x)$ satisfy condition (i) of Theorem 1. Mao *et al.* (2000) show that a suitable Lyapunov function for this system is

$$V(x) = \Sigma_{i=1}^{n} c_i \bar{x}_i h\left(x_i/\bar{x}_i\right) \qquad \text{where} \qquad h(u) = u - 1 - \ln(u), \qquad (6.1)$$

for n positive constants $c_1, ..., c_n$. This function satisfies conditions (ii) and (iv) of Theorem 1, whilst it is straightforward to show that

$$LV(x) = -\frac{1}{2}(x(t) - \bar{x})^T H(x(t) - \bar{x}) \tag{6.2}$$

where

$$H = -CA - A^T C - \sigma^T \mathrm{diag}((c_1\bar{x}_1, ..., c_n\bar{x}_n)\sigma \tag{6.3}$$

and $C = \mathrm{diag}(c_1, ..., c_n)$. Thus, if the n positive constants $c_1, ..., c_n$ can be found such that the symmetric matrix H is non-negative definite then it follows that $LV(x) \leq 0$ and condition (iii) of Theorem 1 is satisfied. Finally, we note that under this condition, Theorem 1 of Mao *et al.* (2000) shows that for any $x_0 \in R^n_+ = \{x \in R^2 : x_i > 0 \text{ for all } 1 \leq i \leq n\}$ $x(t) \in R^n_+$ for all $t \geq 0$ almost surely, and furthermore that $x(t)$ is unique. Therefore, when H is non-negative definite for any $x_0 \in R^2_+$, the Euler scheme will converge to the true solution of (1.5) in the sense of Theorem 1, provided that the time step Δt is sufficiently small. Mao *et al.* (2000) make extensive use of simulations of (1.5) to confirm analytic results and to explore model behaviour in regimes not amenable to analysis. Since these simulations are based on the Euler scheme the results of this section support the numerical results in that earlier paper.

6.2 The Quadratic Autocatylator

Many catalytic systems can be formulated in terms of S.D.E.s. One such example is the quadratic autocatylator which models the catalytic reaction

$$\mathrm{A} + \mathrm{B} \rightarrow 2\mathrm{B}.$$

If these reactions occur in a large volume, in well mixed conditions and in the absence of environmental noise, then the concentrations of A and B particles, $\alpha(t)$ and $\beta(t)$, may be described by the deterministic equations (Marion *et al.* 2000)

$$\begin{aligned} d\alpha(t)/dt &= (\alpha_0 - \alpha(t))\nu - \kappa\alpha(t)\beta(t) \\ d\beta(t)/dt &= \kappa\alpha(t)\beta(t) - (K_b + \nu)\beta(t) \end{aligned}$$

where the parameters α_0, κ, K_b, $\nu \geq 0$. There are many ways in which environmental noise may be introduced into this system, but one simple approach is to assume that the parameter $K_b \rightarrow K_b + \sigma\dot{B}(t)$ is perturbed by noise. This gives rise to the stochastic differential system

$$\begin{aligned} d\alpha(t) &= [(\alpha_0 - \alpha(t))\nu - \kappa\alpha(t)\beta(t)]dt \\ d\beta(t) &= [\kappa\alpha(t)\beta(t) - (K_b + \nu)\beta(t)]dt + \sigma\beta(t)dB(t). \end{aligned} \tag{6.4}$$

The drift and diffusion terms satisfy condition (i) of Theorem 1. In this case define

$$V(\alpha,\beta) = h(\alpha) + h(\beta) \qquad \text{where, as before,} \qquad h(u) = u - 1 - \ln(u). \quad (6.5)$$

Then it is straightforward to see that for $(\alpha,\beta) \in R_+^2$, $V(\alpha,\beta)$ satisfies condition (iii) of theorem 1 since,

$$\begin{aligned} LV(\alpha,\beta) &\le \left(\nu\alpha_0 + 2\nu + K_b + \sigma^2/2\right) + k\beta \\ &\le K(1 + V(\alpha,\beta)). \end{aligned}$$

Moreover, (6.5) also satisfies conditions (ii) and (iv) of Theorem 1. So in order to prove convergence it is sufficient to show that for any $(\alpha(0),\beta(0)) \in R_+^2$ the solution $(\alpha(t),\beta(t))$ remains in R_+^2 for $t \in [0,T]$ for all $T > 0$. To achieve this we note that the limit, as $r \to \infty$, of

$$\mathcal{D}(r) \equiv \{(\alpha,\beta) \in R^2 \textit{ such that } V(\alpha,\beta) \le r\},$$

is $\lim_{r\to\infty} \mathcal{D}(r) = R_+^2$. Thus, applying Lemma 4 we find that for finite T, $t \in [0,T]$ and $(\alpha(0),\beta(0)) \in R_+^2$ the solution $(\alpha(t),\beta(t))$ remains in R_+^2 and is unique. So, the Euler approximate solution will converge to the true solution of (6.4) for any $(\alpha(0),\beta(0)) \in R_+^2$ in the sense of theorem 1, provided Δt is sufficiently small. Marion *et al.* (2000) employ the Euler scheme to study auto-catalytic systems of the type considered here. The results of this paper support this numerical approach.

7 Discussion

We have shown that under the conditions of Theorem 1 the Euler approximate solutions will converge to the true solutions of (1.1) with large probability, provided that the time step is sufficiently small. Furthermore, under the same conditions Lemma 4 identifies an invariant set of the stochastic differential equation (1.1) and shows that its solutions are unique up to any finite time. However, there are a number of ways in which one might improve upon these results. One possible extension would be to widen the applicability of theorem 1 to a non-autonomous system

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t).$$

However, the key difficulty in applying Theorem 1 is in identifying a function V appropriate to a given system and this might become considerably easier if less stringent conditions could be imposed on V. Finally, whilst we have bounded the order of approximation achieved by the Euler scheme it would also be very useful to obtain a more accurate estimate of the approximation error. We hope to address these issues in subsequent work.

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